Problem Set 2 due March 4, at 10 PM, on Gradescope

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue.

Problem 1:

(1) Prove, by direct computation, that if:

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then:

$$X^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The number $ad - bc$ must be invertible (i.e. non-zero) for X^{-1} to make sense. *(10 points)*

Solution: We need to explicitly show that $XX^{-1} = I_2$. Indeed, we have:

$$XX^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & da - cb \end{bmatrix} = I_2$$

Grading Rubric:

- Correct computation *(10 points)*
- Small errors *(6-8 points)*
- Big errors, but matrix multiplication is generally done well *(3-5 points)*
- No answer, or significant misunderstanding of how matrix multiplication works *(0 points)*

Please give at least 3 points to any student who knew they're supposed to show $XX^{-1} = I$.

(2) Assume A, B, C, D are all 2×2 matrices, and assume that:

$$AD - BC = DA - CB = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \qquad \text{and} \qquad AB = BA \qquad \text{and} \qquad CD = DC$$

for some number $\lambda \neq 0$. Under these assumptions, what is the inverse of the block 4×4 matrix:

$$X = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Prove your formula for X^{-1} by showing all the steps. *(10 points)*

Solution: By analogy with part (1), we guess that the correct formula for the inverse is:

$$X^{-1} = \frac{1}{\lambda}\left[\begin{array}{c|c} D & -B \\ \hline -C & A \end{array}\right]$$

To prove that this is the correct solution, we must show that $XX^{-1} = I_4$, and we will do so by matrix multiplication with the blocks:

$$XX^{-1} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \frac{1}{\lambda}\left[\begin{array}{c|c} D & -B \\ \hline -C & A \end{array}\right] = \frac{1}{\lambda}\left[\begin{array}{c|c} AD-BC & -AB+BA \\ \hline CD-DC & -CB+DA \end{array}\right]$$

By assumption, the blocks of the matrix on the right are $\lambda \cdot I_2$, 0, 0, $\lambda \cdot I_2$, so we conclude that:

$$XX^{-1} = \frac{1}{\lambda}\left[\begin{array}{c|c} \lambda \cdot I_2 & 0 \\ \hline 0 & \lambda \cdot I_2 \end{array}\right] = I_4$$

Grading Rubric:

- Correct computation by showing all the steps *(10 points)*
- Skipped steps and did not use the assumptions, or took the commutativity of A, B or C, D for granted *(5 points)*
- Gave the correct formula, but with little or no justification *(5 points)*
- No answer, or significantly incorrect answer and computation. *(0 points)*

Problem 2:

(1) Compute the inverses of the matrices:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{bmatrix}$$

for various numbers a, b, c, a', b', c'. **You must use the Gauss-Jordan elimination procedure** outlined on page 13 of the Lecture notes (or page 86 of the textbook), that is, by starting from the augmented matrices $[\ L \mid I\ ]$ and $[\ U \mid I\ ]$. *(10 points)*

Solution: We perform the Gauss-Jordan elimination procedure for L (we'll write $r_i - \alpha r_j$ to indicate that we subtract α times the j–th row from the i–th row):

$$[\ L \mid I\ ] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ a & 1 & 0 & 0 & 1 & 0 \\ c & b & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 - ar_1} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ c & b & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 - cr_1 - br_2} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 1 & ab-c & -b & 1 \end{bmatrix},$$

so

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab-c & -b & 1 \end{bmatrix}$$

Similarly, for U:

$$\left[\ U \mid I\ \right] = \begin{bmatrix} 1 & a' & c' & 1 & 0 & 0 \\ 0 & 1 & b' & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 - b' r_3} \begin{bmatrix} 1 & a' & c' & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -b' \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 - a' r_2 - c' r_3} \begin{bmatrix} 1 & 0 & 0 & 1 & -a' & a'b' - c' \\ 0 & 1 & 0 & 0 & 1 & -b' \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

so

$$U^{-1} = \begin{bmatrix} 1 & -a' & a'b' - c' \\ 0 & 1 & -b' \\ 0 & 0 & 1 \end{bmatrix}.$$

Grading Rubric for each of the two matrices:

- Correct computation by showing all the steps *(5 points)*
- Small algebra errors *(3 points)*
- Significant misunderstanding of Gauss-Jordan elimination *(0 points)*

(2) Compute the inverse of the matrix:

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

for any three non-zero numbers d_1, d_2, d_3. *(5 points)*

Solution:

$$D^{-1} = \begin{bmatrix} d_1^{-1} & 0 & 0 \\ 0 & d_2^{-1} & 0 \\ 0 & 0 & d_3^{-1} \end{bmatrix}.$$

Grading Rubric:

- Correct matrix *(5 points)*
- Incorrect matrix *(0 points)*

(3) Consider the 3×3 matrix $A = LDU$, where L, D and U are as above. Write A^{-1} as a product of three matrices, and as a single matrix. Note that the formula for A^{-1} as a single matrix will not be particularly pretty, but it will give you some practice with matrix multiplication. *(10 points)*

Solution: For invertible matrices M_1, M_2 of the same size, their product $M_1 M_2$ is invertible since:

$$(M_1 M_2)^{-1} = M_2^{-1} M_1^{-1}$$

Hence:

$$A^{-1} = (LDU)^{-1} = U^{-1} D^{-1} L^{-1} \tag{1}$$

As a single matrix, we may use the formulas we developed in parts (1) and (2) to get:

$$\begin{bmatrix} 1 & -a' & a'b' - c' \\ 0 & 1 & -b' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1^{-1} & 0 & 0 \\ 0 & d_2^{-1} & 0 \\ 0 & 0 & d_3^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab - c & -b & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{d_1} & -\frac{a'}{d_2} & \frac{a'b' - c'}{d_3} \\ 0 & \frac{1}{d_2} & -\frac{b'}{d_3} \\ 0 & 0 & \frac{1}{d_3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab - c & -b & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{d_1} + \frac{aa'}{d_2} + \frac{(a'b'-c')(ab-c)}{d_3} & -\frac{a'}{d_2} - \frac{b(a'b'-c')}{d_3} & \frac{a'b'-c'}{d_3} \\ -\frac{a}{d_2} - \frac{b'(ab-c)}{d_3} & \frac{1}{d_2} + \frac{bb'}{d_3} & -\frac{b'}{d_3} \\ \frac{ab-c}{d_3} & \frac{-b}{d_3} & \frac{1}{d_3} \end{bmatrix}$$

Grading Rubric:

- Got formula (1) and obtained the correct explicit formula for A^{-1} *(10 points)*
- Got formula (1) and obtained an explicit formula for A^{-1} with small mistakes *(7-8 points)*
- Got formula (1) and obtained a significantly wrong formula for A^{-1} *(5 points)*
- Computed A^{-1}, but without formula (1) *(5 points)*
- Didn't get (1) nor the formula for A^{-1} *(0 points)*

Please give 3 points if students started from a wrong formula analogous to (1), e.g. $A^{-1} = L^{-1}D^{-1}U^{-1}$, but computed correctly afterwards.

Problem 3:

(1) Gaussian elimination involves doing row operations on matrices. But what if instead one did column operations on a matrix? For example, start with:

$$A = \begin{bmatrix} \boxed{-1} & -3 & 2 \\ 2 & \boxed{\boxed{7}} & 5 \\ -1 & 0 & 3 \end{bmatrix}$$

and add appropriate multiples of the first column to the second and third columns, so that all entries to the right of the box vanish. Then add an appropriate multiple of the second column to the third column, so that all entries to the right of the double box vanish. Carry out this process by showing all the steps. *(10 points)*

Solution: We will write $c_i - \alpha c_j$ to indicate that we subtract α times the j–th column from the i–th column. Then we have:

$$\begin{bmatrix} \boxed{-1} & -3 & 2 \\ 2 & \boxed{\boxed{7}} & 5 \\ -1 & 0 & 3 \end{bmatrix} \xrightarrow{c_2 - 3c_1} \begin{bmatrix} \boxed{-1} & 0 & 2 \\ 2 & \boxed{\boxed{1}} & 5 \\ -1 & 3 & 3 \end{bmatrix} \xrightarrow{c_3 + 2c_1} \begin{bmatrix} \boxed{-1} & 0 & 0 \\ 2 & \boxed{\boxed{1}} & 9 \\ -1 & 3 & 1 \end{bmatrix} \xrightarrow{c_3 - 9c_2} \begin{bmatrix} \boxed{-1} & 0 & 0 \\ 2 & \boxed{\boxed{1}} & 0 \\ -1 & 3 & -26 \end{bmatrix}$$

Grading Rubric:

- Correct process and answer *(10 points)*
- Correct process but a wrong computation somewhere *(7 points)*
- Correct process but did rows instead of columns *(3 points)*
- Incorrect process *(0 points)*

(2) Describe each of the steps in the process above as multiplying A on the right with an appropriate matrix. What factorization of A does this produce? *(10 points)*

Solution: The first step corresponds to multiplication on the right by:

$$E_{12}^{(-3)} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the second step corresponds to multiplication on the right by:

$$E_{13}^{(2)} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the third step corresponds to multiplication on the right by:

$$E_{23}^{(-9)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -9 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, we conclude that:

$$AE_{12}^{(-3)}E_{13}^{(2)}E_{23}^{(-9)} = L$$

where L is lower triangular. We can invert the elimination matrices above, and get the factorization:

$$A = L\underbrace{E_{23}^{(9)}E_{13}^{(-2)}E_{12}^{(3)}}_{\text{call this } U}$$

where U is upper triangular with 1's on the diagonal. We still get a version of the LU factorization.

Grading Rubric:

- Correct process and answer *(10 points)*
- Did not identify the correct elimination matrices or had them multiply on the left (note: if the student got the wrong elimination matrices just because they made an algebra mistake in part (1), accept the mistake in this part) *(5 points)*
- Did not notice that this leads to an LU factorization, where L is lower triangular and U is upper triangular (irrespective of the entries on the diagonal) *(5 points)*
- Did not do either of the two bullets above *(0 points)*

If students got the product of elimination matrices in the wrong order, please subtract 2 points.

Problem 4:

(1) Consider the matrix:

$$\begin{bmatrix} 2 & 1 & 0 \\ 5 & 4 & 3 \\ 4 & 7 & 6 \end{bmatrix}$$

and write it as the sum of a symmetric matrix and an anti-symmetric matrix (recall that while a symmetric matrix is one such that $S = S^T$, an anti-symmetric matrix is one such that $A = -A^T$). *(10 points)*

Solution: Let

$$S = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 4 & 5 \\ 2 & 5 & 6 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 0 & -2 & -2 \\ 2 & 0 & -2 \\ 2 & 2 & 0 \end{bmatrix}.$$

Grading Rubric (for each one of the two matrices):

- Correct answer *(5 points)*
- -1 point per algebra error
- No answer *(0 points)*

(2) For a general matrix X, suppose you want to write it as $X = S + A$, where S is symmetric and A is anti-symmetric. Can you find formulas for S and A in terms of X only? *(5 points)*

Solution: Let:

$$S = \frac{X + X^T}{2} \qquad \text{and} \qquad A = \frac{X - X^T}{2}$$

Then, clearly $S + A = X$, and $S^T = \frac{X^T + (X^T)^T}{2} = \frac{X^T + X}{2} = S$ and $A^T = \frac{X^T - (X^T)^T}{2} = \frac{X^T - X}{2} = -A$.

Grading Rubric (for each one of the two matrices):

- Correct answer and explanation of why the explicit formulas for S and A in terms of X are symmetric and anti-symmetric, respectively. *(5 points)*
- Correct answer but no explanation *(3 points)*
- Incorrect answer *(0 points)*

Problem 5:

(1) Consider the matrix:

$$\begin{bmatrix} 1 & 5 & 0 & 0 \\ 1 & 7 & 6 & 0 \\ 0 & 4 & 15 & 7 \\ 0 & 0 & 9 & 25 \end{bmatrix}$$

and compute its LU factorization. *(10 points)*

Solution: We first perform Gaussian elimination on the matrix:

$$\begin{bmatrix}1&5&0&0\\1&7&6&0\\0&4&15&7\\0&0&9&25\end{bmatrix}\xrightarrow{r_2-r_1}\begin{bmatrix}1&5&0&0\\0&2&6&0\\0&4&15&7\\0&0&9&25\end{bmatrix}\xrightarrow{r_3-2r_2}\begin{bmatrix}1&5&0&0\\0&2&6&0\\0&0&3&7\\0&0&9&25\end{bmatrix}\xrightarrow{r_4-3r_3}\begin{bmatrix}1&5&0&0\\0&2&6&0\\0&0&3&7\\0&0&0&4\end{bmatrix},$$

which expressed as matrix operations corresponds to the equation:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-3&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-2&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\-1&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&5&0&0\\1&7&6&0\\0&4&15&7\\0&0&9&25\end{bmatrix}=\begin{bmatrix}1&5&0&0\\0&2&6&0\\0&0&3&7\\0&0&0&4\end{bmatrix},$$

so

$$\begin{bmatrix}1&5&0&0\\1&7&6&0\\0&4&15&7\\0&0&9&25\end{bmatrix}=\begin{bmatrix}1&0&0&0\\-1&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-2&1&0\\0&0&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-3&1\end{bmatrix}^{-1}\begin{bmatrix}1&5&0&0\\0&2&6&0\\0&0&3&7\\0&0&0&4\end{bmatrix},$$

or

$$\begin{bmatrix}1&5&0&0\\1&7&6&0\\0&4&15&7\\0&0&9&25\end{bmatrix}=\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&3&1\end{bmatrix}\begin{bmatrix}1&5&0&0\\0&2&6&0\\0&0&3&7\\0&0&0&4\end{bmatrix}.$$

Thus, since:

$$\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&3&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&2&1&0\\0&0&3&1\end{bmatrix},$$

we conclude that the LU factorization of the matrix we started with is:

$$L=\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&2&1&0\\0&0&3&1\end{bmatrix}\qquad U=\begin{bmatrix}1&5&0&0\\0&2&6&0\\0&0&3&7\\0&0&0&4\end{bmatrix}$$

Grading Rubric (for each one of the two matrices:

- Correct *(10 points)*
- Minor computational errors *(6-8 points)*
- Significant computational errors *(3-5 points)*
- No answer or wrong method of Gaussian elimination *(0 points)*

(2) Based on the example in part (1), how do you think the LU factorization of a matrix of the form:

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 & 0 & 0 \\ 0 & 0 & 0 & a_5 & b_5 & c_5 & 0 \\ 0 & 0 & 0 & 0 & a_6 & b_6 & c_6 \\ 0 & 0 & 0 & 0 & 0 & a_7 & b_7 \end{bmatrix}$$

will look like? (Just a general guess on how the matrices L and U will look will suffice for now. Hint: L and U will have a lot of zeroes. Where do you think they are located?) *(5 points)*

Solution: The L matrix will have 1's along the main diagonal, possibly nonzero entries on the subdiagonal, and all other entries 0. The U matrix will have possibly non-zero entries on the diagonal and superdiagonal, and all other entries 0.

Grading Rubric (for each one of the two matrices:

- Full description of the L and U matrices, including the fact that all entries away from the diagonal/subdiagonal/superdiagonal are 0 (students don't need to use this specific terminology) *(5 points)*
- Students mention that U is upper and L is lower triangular *(2 points)*
- None of the above *(0 points)*

(3) In the generality of part (2), what are the entries of U above the diagonal, and why? *(5 points)*

Solution: Above the diagonal, the matrix U is the same as the original matrix, i.e.:

$$U = \begin{bmatrix} * & c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & c_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & c_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

This is because when you multiply such a matrix with L, which is a lower triangular matrix with 1's on the diagonal:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 1 \end{bmatrix}$$

the entries above the diagonal do not change (i.e. they are the same in U as in LU).

- An explanation morally equivalent to the one above *(5 points)*
- Just a statement of the correct answer for the entries of U above the diagonal, but without explanation *(3 points)*
- None of the above *(0 points)*